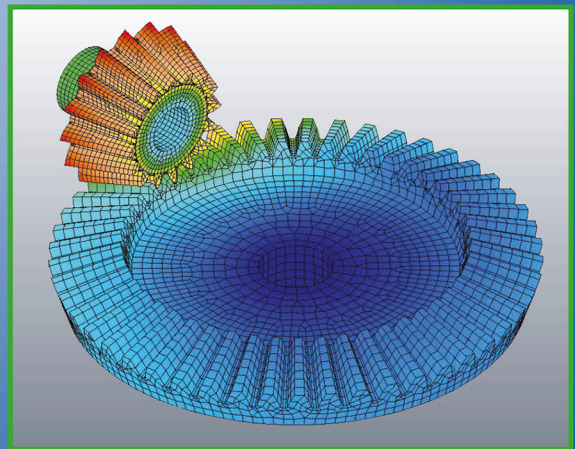
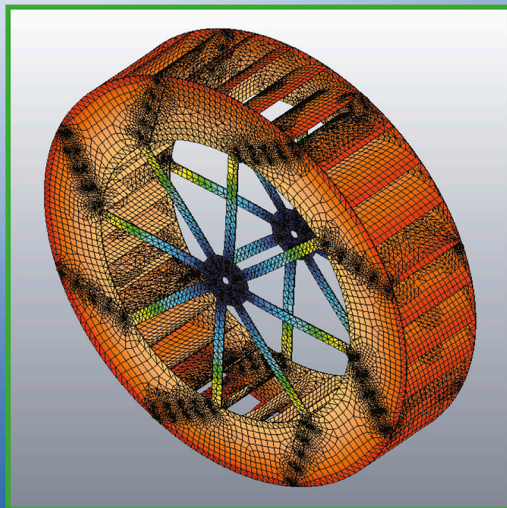
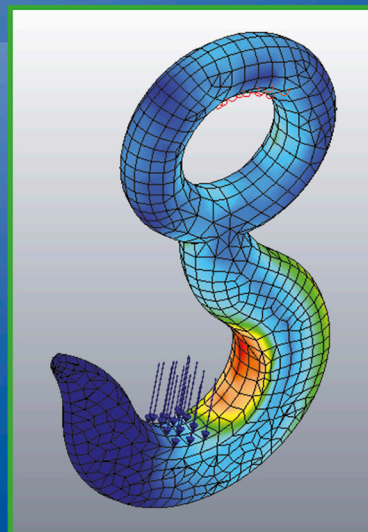


Sixth Edition

A First Course in the
Finite Element Method



DARYL L. LOGAN



CONVERSION FACTORS U.S. Customary Units to SI Units

Quantity Converted from U.S. Customary	To	SI Equivalent
(Acceleration)		
1 foot/second ² (ft/s ²)	meter/second ² (m/s ²)	0.3048 m/s ²
1 inch/second ² (in./s ²)	meter/second ² (m/s ²)	0.0254 m/s ²
(Area)		
1 foot ² (ft ²)	meter ² (m ²)	0.0929 m ²
1 inch ² (in. ²)	meter ² (m ²)	645.2 mm ²
(Density, mass)		
1 pound mass/inch ³ (lbm/in. ³)	kilogram/meter ³ (kg/m ³)	27.68 Mg/m ³
1 pound mass/foot ³ (lbm/ft ³)	kilogram/meter ³ (kg/m ³)	16.02 kg/m ³
(Energy, Work)		
1 British thermal unit (BTU)	Joule (J)	1055 J
1 foot-pound force (ft-lb)	Joule (J)	1.356 J
1 kilowatt-hour	Joule (J)	3.60 × 10 ⁶ J
(Force)		
1 kip (1000 lb)	Newton (N)	4.448 kN
1 pound force (lb)	Newton (N)	4.448 N
(Length)		
1 foot (ft)	meter (m)	0.3048 m
1 inch (in.)	meter (m)	25.4 mm
1 mile (mi), (U.S. statute)	meter (m)	1.609 km
1 mile (mi), (international nautical)	meter (m)	1.852 km
(Mass)		
1 pound mass (lbm)	kilogram (kg)	0.4536 kg
1 slug (lb-sec ² /ft)	kilogram (kg)	14.59 kg
1 metric ton (2000 lbm)	kilogram (kg)	907.2 kg
(Moment of force)		
1 pound-foot (lb·ft)	Newton-meter (N·m)	1.356 N·m
1 pound-inch (lb·in.)	Newton-meter (N·m)	0.1130 N·m
(Moment of inertia of an area)		
1 inch ⁴	meter ⁴ (m ⁴)	0.4162 × 10 ⁻⁶ m ⁴
(Moment of inertia of a mass)		
1 pound-foot-second ² (lb·ft·s ²)	kilogram-meter ² (kg·m ²)	1.356 kg·m ²
(Momentum, linear)		
1 pound-second (lb·s)	kilogram-meter/second (kg·m/s)	4.448 N·s
(Momentum, angular)		
pound-foot-second (lb·ft·s)	Newton-meter-second (N·m·s)	1.356 N·m·s

CONVERSION FACTORS U.S. Customary Units to SI Units (Continued)

Quantity Converted from U.S. Customary	To	SI Equivalent
(Power)		
1 foot-pound/second (ft·lb/s)	Watt (W)	1.356 W
1 horsepower (550 ft·lb/s)	Watt (W)	745.7 W
(Pressure, stress)		
1 atmosphere (std)(14.7.lb/in. ²)	Newton/meter ² (N/m ² or Pa)	101.3 kPa
1 pound/foot ² (lb/ft ²)	Newton/meter ² (N/m ² or Pa)	47.88 Pa
1 pound/inch ² (lb/in. ² or psi)	Newton/meter ² (N/m ² or Pa)	6.895 kPa
1 kip/inch ² (ksi)	Newton/meter ² (N/m ² or Pa)	6.895 MPa
(Spring constant)		
1 pound/inch (lb/in.)	Newton/meter (N/m)	175.1 N/m
(Temperature)		
$T(^{\circ}\text{F}) = 1.8T(^{\circ}\text{C}) + 32$		
(Velocity)		
1 foot/second (ft/s)	meter/second (m/s)	0.3048 m/s
1 knot (nautical mi/h)	meter/second (m/s)	0.5144 m/s
1 mile/hour (mi/h)	meter/second (m/s)	0.4470 m/s
1 mile/hour (mi/h)	kilometer/hour (km/h)	1.609 km/h
(Volume)		
1 foot ³ (ft ³)	meter ³ (m ³)	0.02832 m ³
1 inch ³ (in. ³)	meter ³ (m ³)	$16.39 \times 10^{-6} \text{ m}^3$

PHYSICAL PROPERTIES IN SI AND USCS UNITS

Property	SI	USCS
Water (fresh)		
specific weight	9.81 kN/m ³	62.4 lb/ft ³
mass density	1000 kg/m ³	1.94 slugs/ft ³
Aluminum		
specific weight	26.6 kN/m ³	169 lb/ft ³
mass density	2710 kg/m ³	5.26 slugs/ft ³
Steel		
specific weight	77.0 kN/m ³	490 lb/ft ³
mass density	7850 kg/m ³	15.2 slugs/ft ³
Reinforced concrete		
specific weight	23.6 kN/m ³	150 lb/ft ³
mass density	2400 kg/m ³	4.66 slugs/ft ³
Acceleration of gravity (on the earth's surface)		
Recommended value	9.81 m/s ²	32.2 ft/s ²
Atmospheric pressure (at sea level)		
Recommended value	101 kPa	14.7 psi

TYPICAL PROPERTIES OF SELECTED ENGINEERING MATERIALS

Material	Ultimate Strength σ_u		0.2% Yield Strength σ_y		Modulus of Elasticity E		Sheer Modulus G	Coefficient of Thermal Expansion, α		Density, ρ	
	ksi	MPa	ksi	MPa	(10 ⁶ psi)	GPa	(10 ⁶ psi)	10 ⁻⁶ /°F	10 ⁻⁶ /°C	lb/in. ³	kg/m ³
Aluminum											
Alloy 1100-H14 (99 % A1)	14	110(T)	14	95	10.1	70	3.7	13.1	23.6	0.098	2710
Alloy 2024-T3 (sheet and plate)	70	480(T)	50	340	10.6	73	4.0	12.6	22.7	0.100	2763
Alloy 6061-T6 (extruded)	42	260(T)	37	255	10.0	69	3.7	13.1	23.6	0.098	2710
Alloy 7075-T6 (sheet and plate)	80	550(T)	70	480	10.4	72	3.9	12.9	23.2	0.101	2795
Yellow brass (65% Cu, 35% Zn)											
Cold-rolled	78	540(T)	63	435	15	105	5.6	11.3	20.0	0.306	8470
Annealed	48	330(T)	15	105	15	105	5.6	11.3	20.0	0.306	8470
Phosphor bronze											
Cold-rolled (510)	81	560(T)	75	520	15.9	110	5.9	9.9	17.8	0.320	8860
Spring-tempered (524)	122	840(T)	—	—	16	110	5.9	10.2	18.4	0.317	8780
Cast iron											
Gray, 4.5%C, ASTM A-48	25 95	170(T) 650(C)	—	—	10	70	4.1	6.7	12.1	0.260	7200
Malleable,											
ASTM A-47	50 90	340(T) 620(C)	33 —	230 —	24	165	9.3	6.7	12.1	0.264	7300

TYPICAL PROPERTIES OF SELECTED ENGINEERING MATERIALS (Continued)

Material	Ultimate Strength σ_u		0.2% Yield Strength σ_y		Modulus of Elasticity E		Sheer Modulus G (10^6 psi)	Coefficient of Thermal Expansion, α		Density, ρ	
	ksi	MPa	ksi	MPa	(10^6 psi)	(GPa)		$10^{-6}/^{\circ}\text{F}$	$10^{-6}/^{\circ}\text{C}$	lb/in. ³	kg/m ³
Copper and its alloys											
CDA 145 copper, hard	48	331(T)	44	303	16	110	6.1	9.9	17.8	0.323	8940
CDA 172 beryllium copper, hard	175	1210(T)	240	965	19	131	7.1	9.4	17.0	0.298	8250
CDA 220 bronze, hard	61	421(T)	54	372	17	117	6.4	10.2	18.4	0.318	8800
CDA 260 brass, hard	76	524(T)	63	434	16	110	6.1	11.1	20.0	0.308	8530
Magnesium alloy											
(8.5% Al)	55	380(T)	40	275	4.5	45	2.4	14.5	26.0	0.065	1800
Monel alloy 400 (Ni-Cu)											
Cold-worked	98	675(T)	85	580	26	180	—	7.7	13.9	0.319	8830
Annealed	80	550(T)	32	220	26	180	—	7.7	13.9	0.319	8830
Steel											
Structural											
(ASTM-A36)	58	400(T)	36	250	29	200	11.5	6.5	11.7	0.284	7860
High-strength low-alloy											
ASTM-A242	70	480(T)	50	345	29	200	11.5	6.5	11.7	0.284	7860
Quenched and tempered alloy											
ASTM-A514	120	825(T)	100	690	29	200	11.5	6.5	11.7	0.284	7860
Stainless, (302)											
Cold-rolled	125	860(T)	75	520	28	190	10.6	9.6	17.3	0.286	7920
Annealed	90	620(T)	40	275	28	190	10.6	9.6	17.3	0.286	7920
Titanium alloy											
(6% Al, 4% V)	130	900(T)	120	825	16.5	114	6.2	5.3	9.5	0.161	4460
Concrete											
Medium strength	4.0	28(C)	—	—	3.5	25	—	5.5	10.0	0.084	2320
High strength	6.0	40(C)	—	—	4.5	30	—	5.5	10.0	0.084	2320
Granite	35	240(C)	—	—	10	69	—	4.0	7.0	0.100	2770
Glass, 98% silica	7	50(C)	—	—	10	69	—	44.0	80.0	0.079	2190
Melamine	6	41(T)	—	—	2.0	13.4	—	17.0	30.0	0.042	1162
Nylon, molded	8	55(T)	—	—	0.3	2	—	45.0	81.0	0.040	1100
Polystyrene	7	48(T)	—	—	0.45	3	—	40.0	72.0	0.038	1050
Rubbers											
Natural	2	14(T)	—	—	—	—	—	90.0	162.0	0.033	910
Neoprene	3.5	24(T)	—	—	—	—	—	—	—	0.045	1250
Timber, air dry, parallel to grain											
Douglas fir, construction											
grade	7.2	50(C)	—	—	1.5	10.5	—	varies	varies	0.019	525
Eastern spruce	5.4	37(C)	—	—	1.3	9	—	1.7–	3–	0.016	440
Southern pine, construction											
grade	7.3	50(C)	—	—	1.2	8.3	—	3.0	5.4	0.022	610

The values given in the table are average mechanical properties. Further verification may be necessary for final design or analysis. For ductile materials, the compressive strength is normally assumed to equal the tensile strength. *Abbreviations:* C, compressive strength; T, tensile strength. For an explanation of the numbers associated with the aluminums, cast irons, and steels, see ASM Metals Reference Book, latest ed., American Society for Metals, Metals Park, Ohio 44073

A First Course in the Finite Element Method

SIXTH EDITION

Daryl L. Logan

University of Wisconsin–Platteville



Australia • Brazil • Mexico • Singapore • United Kingdom • United States

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PREFACE

Features and Approach

The purpose of this sixth edition is to provide an introductory approach to the finite element method that can be understood by both undergraduate and graduate students without the usual prerequisites (such as structural analysis and upper level calculus) required by many available texts in this area. The book is written primarily as a basic learning tool for the undergraduate student in civil and mechanical engineering whose main interest is in stress analysis and heat transfer, although material on fluid flow in porous media and through hydraulic networks and electrical networks and electrostatics is also included. The concepts are presented in sufficiently simple form with numerous example problems logically placed throughout the book, so that the book serves as a valuable learning aid for students with other backgrounds, as well as for practicing engineers. The text is geared toward those who want to apply the finite element method to solve practical physical problems.

General principles are presented for each topic, followed by traditional applications of these principles, including longhand solutions, which are in turn followed by computer applications where relevant. The approach is taken to illustrate concepts used for computer analysis of large-scale problems.

The book proceeds from basic to advanced topics and can be suitably used in a two-course sequence. Topics include basic treatments of (1) simple springs and bars, leading to two- and three-dimensional truss analysis; (2) beam bending, leading to plane frame, grid, and space frame analysis; (3) elementary plane stress/strain elements, leading to more advanced plane stress/strain elements and applications to more complex plane stress/strain analysis; (4) axisymmetric stress analysis; (5) isoparametric formulation of the finite element method; (6) three-dimensional stress analysis; (7) plate bending analysis; (8) heat transfer and fluid mass transport; (9) basic fluid flow through porous media and around solid bodies, hydraulic networks, electric networks, and electrostatics analysis; (10) thermal stress analysis; and (11) time-dependent stress and heat transfer.

Additional features include how to handle inclined or skewed boundary conditions, beam element with nodal hinge, the concept of substructure, the patch test, and practical considerations in modeling and interpreting results.

The direct approach, the principle of minimum potential energy, and Galerkin's residual method are introduced at various stages, as required, to develop the equations needed for analysis.

Appendices provide material on the following topics: (A) basic matrix algebra used throughout the text; (B) solution methods for simultaneous equations; (C) basic theory of elasticity; (D) work-equivalent nodal forces; (E) the principle of virtual work; and (F) properties of structural steel shapes.

More than 100 solved examples appear throughout the text. Most of these examples are solved “longhand” to illustrate the concepts. More than 570 end-of-chapter problems are provided to reinforce concepts. The answers to many problems are included in the back of the book to aid those wanting to verify their work. Those end-of-chapter problems to be solved using a computer program are marked with a computer symbol.

Additional Features

Additional features of this edition include updated notation used by most engineering instructors, chapter objectives at the start of each chapter to help students identify what content is most important to focus on and retain summary equations for handy use at the end of each chapter, additional information on modeling, and more comparisons of finite element solutions to analytical solutions.

New Features

Over 140 new problems for solution have been included, and additional design-type problems have been added to chapters 3, 5, 7, 11, and 12. Additional real-world applications from industry have been added to enhance student understanding and reinforce concepts. New space frames, solid-model-type examples, and problems for solution have been added. New examples from other fields now demonstrate how students can use the Finite Element Method to solve problems in a variety of engineering and mathematical physics areas. As in the 5th edition, this edition deliberately leaves out consideration of special purpose computer programs and suggests that instructors choose a program they are familiar with to integrate into their finite element course.

Resources for Instructors

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Suggested Topics

Following is an outline of suggested topics for a first course (approximately 44 lectures, 52 minutes each) in which this textbook is used.

Topic	Number of Lectures
Appendix A	1
Appendix B	1
Chapter 1	2
Chapter 2	3
Chapter 3, Sections 3.1–3.11, 3.14 and 3.15	5
Exam 1	1
Chapter 4, Sections 4.1–4.6	4
Chapter 5, Sections 5.1–5.3, 5.5	4
Chapter 6	4
Chapter 7	3
Exam 2	1
Chapter 9	2
Chapter 10	4
Chapter 11	3
Chapter 13, Sections 13.1–13.7	5
Chapter 15	3
Exam 3	1

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NOTATION

English Symbols

a_i	generalized coordinates (coefficients used to express displacement in general form)
A	cross-sectional area
$[B]$	matrix relating strains to nodal displacements or relating temperature gradient to nodal temperatures
c	specific heat of a material
$[C']$	matrix relating stresses to nodal displacements
C	direction cosine in two dimensions
C_x, C_y, C_z	direction cosines in three dimensions
$\{d\}$	element and structure nodal displacement matrix, both in global coordinates
$\{d'\}$	local-coordinate element nodal displacement matrix
D	bending rigidity of a plate
$[D]$	matrix relating stresses to strains
$[D']$	operator matrix given by Eq. (10.2.16)
e	exponential function
E	modulus of elasticity
$\{f\}$	global-coordinate nodal force matrix
$\{f'\}$	local-coordinate element nodal force matrix
$\{f_b\}$	body force matrix
$\{f_h\}$	heat transfer force matrix
$\{f_q\}$	heat flux force matrix
$\{f_Q\}$	heat source force matrix
$\{f_s\}$	surface force matrix
$\{F\}$	global-coordinate structure force matrix
$\{F_c\}$	condensed force matrix
$\{F_i\}$	global nodal forces
$\{F_0\}$	equivalent force matrix
$\{g\}$	temperature gradient matrix or hydraulic gradient matrix
G	shear modulus
h	heat-transfer (or convection) coefficient
i, j, m	nodes of a triangular element
I	principal moment of inertia
$[J]$	Jacobian matrix
k	spring stiffness
$[k]$	global-coordinate element stiffness or conduction matrix
$[k_c]$	condensed stiffness matrix, and conduction part of the stiffness matrix in heat-transfer problems
$[k']$	local-coordinate element stiffness matrix

$[k_n]$	convective part of the stiffness matrix in heat-transfer problems
$[K]$	global-coordinate structure stiffness matrix
K_{xx}, K_{yy}	thermal conductivities (or permeabilities, for fluid mechanics) in the x and y directions, respectively
L	length of a bar or beam element
m	maximum difference in node numbers in an element
$m(x)$	general moment expression
m_x, m_y, m_{xy}	moments in a plate
$[m']$, $[m]$	local element mass matrix
$[m'_i]$	local nodal moments
$[M]$	global mass matrix
$[M^*]$	matrix used to relate displacements to generalized coordinates for a linear-strain triangle formulation
$[M']$	matrix used to relate strains to generalized coordinates for a linear-strain triangle formulation
n_b	bandwidth of a structure
n_d	number of degrees of freedom per node
$[N]$	shape (interpolation or basis) function matrix
N_i	shape functions
p	surface pressure (or nodal heads in fluid mechanics)
p_r, p_z	radial and axial (longitudinal) pressures, respectively
P	concentrated load
$[P']$	concentrated local force matrix
q	heat flow (flux) per unit area or distributed loading on a plate
\bar{q}	rate of heat flow
q^*	heat flow per unit area on a boundary surface
Q	heat source generated per unit volume or internal fluid source
Q^*	line or point heat source
Q_x, Q_y	transverse shear line loads on a plate
r, θ, z	radial, circumferential, and axial coordinates, respectively
R	residual in Galerkin's integral
R_b	body force in the radial direction
R_{ix}, R_{iy}	nodal reactions in x and y directions, respectively
s, t, z'	natural coordinates attached to isoparametric element
S	surface area
t	thickness of a plane element or a plate element
t_i, t_j, t_m	nodal temperatures of a triangular element
T	temperature function
T_∞	free-stream temperature
$[T]$	displacement, force, and stiffness transformation matrix
$[T_i]$	surface traction matrix in the i direction
u, v, w	displacement functions in the $x, y,$ and z directions, respectively
u_i, v_i, w_i	$x, y,$ and z displacements at node i , respectively

U	strain energy
ΔU	change in stored energy
v	velocity of fluid flow
V	shear force in a beam
w	distributed loading on a beam or along an edge of a plane element
W	work
x_i, y_i, z_i	nodal coordinates in the x , y , and z directions, respectively
x', y', z'	local element coordinate axes
x, y, z	structure global or reference coordinate axes
$[X]$	body force matrix
X_b, Y_b	body forces in the x and y directions, respectively
Z_b	body force in longitudinal direction (axisymmetric case) or in the z direction (three-dimensional case)

Greek Symbols

α	coefficient of thermal expansion
$\alpha_i, \beta_i, \gamma_i, \delta_i$	used to express the shape functions defined by Eq. (6.2.10) and Eqs. (11.2.5) through (11.2.8)
δ	spring or bar deformation
ϵ	normal strain
$\{\epsilon_T\}$	thermal strain matrix
k_x, k_y, k_{xy}	curvatures in plate bending
ν	Poisson's ratio
ϕ_i	nodal angle of rotation or slope in a beam element
π_h	functional for heat-transfer problem
π_p	total potential energy
ρ	mass density of a material
ρ_w	weight density of a material
ω	angular velocity and natural circular frequency
Ω	potential energy of forces
ϕ	fluid head or potential, or rotation or slope in a beam
σ	normal stress
$\{\sigma_T\}$	thermal stress matrix
τ	shear stress and period of vibration
θ	angle between the x axis and the local x' axis for two-dimensional problems
θ_p	principal angle
$\theta_x, \theta_y, \theta_z$	angles between the global x , y , and z axes and the local x' axis, respectively, or rotations about the x and y axes in a plate
$[\psi]$	general displacement function matrix

Other Symbols

$\frac{d(\)}{dx}$	derivative of a variable with respect to x
dt	time differential
$(\dot{\ })$	the dot over a variable denotes that the variable is being differentiated with respect to time
$[\]$	denotes a rectangular or a square matrix
$\{ \}$	denotes a column matrix
$(\underline{\ })$	the underline of a variable denotes a matrix
(\prime)	the prime next to a variable denotes that the variable is being described in a local coordinate system
$[\]^{-1}$	denotes the inverse of a matrix
$[\]^T$	denotes the transpose of a matrix
$\frac{\partial(\)}{\partial x}$	partial derivative with respect to x
$\frac{\partial(\)}{\partial \{x\}}$	partial derivative with respect to each variable in $\{d\}$

Introduction

CHAPTER OBJECTIVES

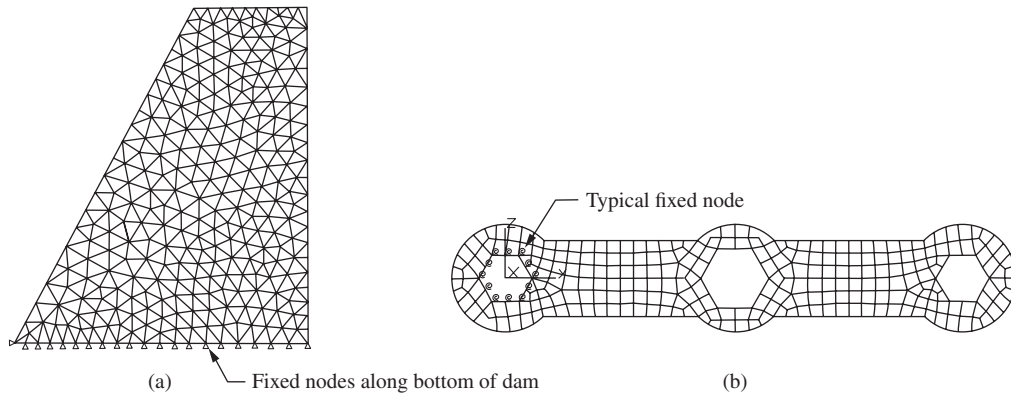
- To present an introduction to the finite element method.
- To provide a brief history of the finite element method.
- To introduce matrix notation.
- To describe the role of the computer in the development of the finite element method.
- To present the general steps used in the finite element method.
- To illustrate the various types of elements used in the finite element method.
- To show typical applications of the finite element method.
- To summarize some of the advantages of the finite element method.

Prologue

The finite element method is a numerical method for solving problems of engineering and mathematical physics. Typical problem areas of interest in engineering and mathematical physics that are solvable by use of the finite element method include structural analysis, heat transfer, fluid flow, mass transport, and electromagnetic potential.

For physical systems involving complicated geometries, loadings, and material properties, it is generally not possible to obtain analytical mathematical solutions to simulate the response of the physical system. Analytical solutions are those given by a mathematical expression that yields the values of the desired unknown quantities at any location in a body (here total structure or physical system of interest) and are thus valid for an infinite number of locations in the body. These analytical solutions generally require the solution of ordinary or partial differential equations, typically created by engineers, physicists, and mathematicians to eliminate the need for the creation and testing of numerous prototype designs, which may be quite costly. Because of the complicated geometries, loadings, and material properties, the solution to these differential equations is usually not obtainable. Hence, we need to rely on numerical methods, such as the finite element method, that can approximate the solution to these equations.

The finite element formulation of the problem results in a system of simultaneous algebraic equations for solution, rather than requiring the solution of differential equations. These numerical



■ **Figure 1–1** Two-dimensional models of (a) discretized dam and (b) discretized bicycle wrench (Applied loads are not shown.) All elements and nodes lie in a plane.

methods yield approximate values of the unknowns at discrete numbers of points in the continuum. Hence, this process of modeling a body by dividing it into an equivalent system of smaller bodies of units (finite elements) interconnected at points common to two or more elements (nodal points or nodes) and/or boundary lines and/or surfaces is called *discretization*. Figure 1–1 shows a cross section of a concrete dam and a bicycle wrench, respectively, that illustrate this process of discretization, where the dam has been divided into 490 plane triangular elements and the wrench has been divided into 254 plane quadrilateral elements. In both models the elements are connected at nodes and along inter element boundary lines. In the finite element method, instead of solving the problem for the entire body in one operation, we formulate the equations for each finite element and then combine them to obtain the solution for the whole body.

Briefly, the solution for structural problems typically refers to determining the displacements at each node and the stresses within each element making up the structure that is subjected to applied loads. In nonstructural problems, the nodal unknowns may, for instance, be temperatures or fluid pressures due to thermal or fluid fluxes.

This chapter first presents a brief history of the development of the finite element method. You will see from this historical account that the method has become a practical one for solving engineering problems only in the past 60 years (paralleling the developments associated with the modern high-speed electronic digital computer). This historical account is followed by an introduction to matrix notation; then we describe the need for matrix methods (as made practical by the development of the modern digital computer) in formulating the equations for solution. This section discusses both the role of the digital computer in solving the large systems of simultaneous algebraic equations associated with complex problems and the development of numerous computer programs based on the finite element method. Next, a general description of the steps involved in obtaining a solution to a problem is provided. This description includes discussion of the types of elements available for a finite element method solution. Various representative applications are then presented to illustrate the capacity of the method to solve problems, such as those involving complicated geometries, several different materials, and irregular loadings. Chapter 1 also lists some of the advantages of the finite element method in solving problems of engineering and mathematical physics. Finally, we present numerous features of computer programs based on the finite element method.

1.1 Brief History

This section presents a brief history of the finite element method as applied to both structural and nonstructural areas of engineering and to mathematical physics. References cited here are intended to augment this short introduction to the historical background.

The modern development of the finite element method began in the 1940s in the field of structural engineering with the work by Hrennikoff [1] in 1941 and McHenry [2] in 1943, who used a lattice of line (one-dimensional) elements (bars and beams) for the solution of stresses in continuous solids. In a paper published in 1943 but not widely recognized for many years, Courant [3] proposed setting up the solution of stresses in a variational form. Then he introduced piecewise interpolation (or shape) functions over triangular subregions making up the whole region as a method to obtain approximate numerical solutions. In 1947 Levy [4] developed the flexibility or force method, and in 1953 his work [5] suggested that another method (the stiffness or displacement method) could be a promising alternative for use in analyzing statically redundant aircraft structures. However, his equations were cumbersome to solve by hand, and thus the method became popular only with the advent of the high-speed digital computer.

In 1954 Argyris and Kelsey [6, 7] developed matrix structural analysis methods using energy principles. This development illustrated the important role that energy principles would play in the finite element method.

The first treatment of two-dimensional elements was by Turner et al. [8] in 1956. They derived stiffness matrices for truss elements, beam elements, and two-dimensional triangular and rectangular elements in plane stress and outlined the procedure commonly known as the *direct stiffness method* for obtaining the total structure stiffness matrix. Along with the development of the high-speed digital computer in the early 1950s, the work of Turner et al. [8] prompted further development of finite element stiffness equations expressed in matrix notation. The phrase *finite element* was introduced by Clough [9] in 1960 when both triangular and rectangular elements were used for plane stress analysis.

A flat, rectangular-plate bending-element stiffness matrix was developed by Melosh [10] in 1961. This was followed by development of the curved-shell bending-element stiffness matrix for axisymmetric shells and pressure vessels by Grafton and Strome [11] in 1963.

Extension of the finite element method to three-dimensional problems with the development of a tetrahedral stiffness matrix was done by Martin [12] in 1961, by Gallagher et al. [13] in 1962, and by Melosh [14] in 1963. Additional three-dimensional elements were studied by Argyris [15] in 1964. The special case of axisymmetric solids was considered by Clough and Rashid [16] and Wilson [17] in 1965.

Most of the finite element work up to the early 1960s dealt with small strains and small displacements, elastic material behavior, and static loadings. However, large deflection and thermal analysis were considered by Turner et al. [18] in 1960 and material nonlinearities by Gallagher et al. [13] in 1962, whereas buckling problems were initially treated by Gallagher and Padlog [19] in 1963. Zienkiewicz et al. [20] extended the method to visco elasticity problems in 1968.

In 1965 Archer [21] considered dynamic analysis in the development of the consistent-mass matrix, which is applicable to analysis of distributed-mass systems such as bars and beams in structural analysis.

With Melosh's [14] realization in 1963 that the finite element method could be set up in terms of a variational formulation, it began to be used to solve nonstructural applications. Field problems, such as determination of the torsion of a shaft, fluid flow, and heat conduction, were solved by Zienkiewicz and Cheung [22] in 1965, Martin [23] in 1968, and Wilson and Nickel [24] in 1966.

Further extension of the method was made possible by the adaptation of weighted residual methods, first by Szabo and Lee [25] in 1969 to derive the previously known elasticity equations used in structural analysis and then by Zienkiewicz and Parekh [26] in 1970 for transient field problems. It was then recognized that when direct formulations and variational formulations are difficult or not possible to use, the method of weighted residuals may at times be appropriate. For example, in 1977 Lyness et al. [27] applied the method of weighted residuals to the determination of magnetic field.

In 1976, Belytschko [28, 29] considered problems associated with large-displacement nonlinear dynamic behavior and improved numerical techniques for solving the resulting systems of equations. For more on these topics, consult the texts by Belytschko, Liu, Moran [58], and Crisfield [61, 62].

A relatively new field of application of the finite element method is that of bioengineering [30, 31]. This field is still troubled by such difficulties as nonlinear materials, geometric nonlinearities, and other complexities still being discovered.

From the early 1950s to the present, enormous advances have been made in the application of the finite element method to solve complicated engineering problems. Engineers, applied mathematicians, and other scientists will undoubtedly continue to develop new applications. For an extensive bibliography on the finite element method, consult the work of Kardstuncer [32], Clough [33], or Noor [57].

1.2 Introduction to Matrix Notation

Matrix methods are a necessary tool used in the finite element method for purposes of simplifying the formulation of the element stiffness equations, for purposes of longhand solutions of various problems, and, most important, for use in programming the methods for high-speed electronic digital computers. Hence matrix notation represents a simple and easy-to-use notation for writing and solving sets of simultaneous algebraic equations.

Appendix A discusses the significant matrix concepts used throughout the text. We will present here only a brief summary of the notation used in this text.

A **matrix** is a rectangular array of quantities arranged in rows and columns that is often used as an aid in expressing and solving a system of algebraic equations. As examples of matrices that will be described in subsequent chapters, the force components $(F_{1x}, F_{1y}, F_{1z}, F_{2x}, F_{2y}, F_{2z}, \dots, F_{nx}, F_{ny}, F_{nz})$ acting at the various nodes or points $(1, 2, \dots, n)$ on a structure and the corresponding set of nodal displacements $(u_1, v_1, w_1, u_2, v_2, w_2, \dots, u_n, v_n, w_n)$ can both be expressed as matrices:

$$\{F\} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{1z} \\ F_{2x} \\ F_{2y} \\ F_{2z} \\ \vdots \\ F_{nx} \\ F_{ny} \\ F_{nz} \end{Bmatrix} \quad \{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix} \quad (1.2.1)$$

The subscripts to the right of F identify the node and the direction of force, respectively. For instance, F_{1x} denotes the force at node 1 applied in the x direction. The x , y , and z displacements at a node are denoted by u , v , and w , respectively. The subscript next to u , v , and w denotes the node. For instance, u_1 , v_1 , and w_1 denote the displacement components in the x , y , and z directions, respectively, at node 1. The matrices in Eqs. (1.2.1) are called *column matrices* and have a size of $n \times 1$. The brace notation $\{\}$ will be used throughout the text to denote a column matrix. The whole set of force or displacement values in the column matrix is simply represented by $\{F\}$ or $\{d\}$.

The more general case of a known rectangular matrix will be indicated by use of the bracket notation $[]$. For instance, the element and global structure stiffness matrices $[k]$ and $[K]$, respectively, developed throughout the text for various element types (such as those in Figure 1–2 on page 11), are represented by square matrices given as

$$[k] = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nm} \end{bmatrix} \quad (1.2.2)$$

and

$$[K] = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nm} \end{bmatrix} \quad (1.2.3)$$

where, in structural theory, the elements k_{ij} and K_{ij} are often referred to as *stiffness influence coefficients*.

You will learn that the global nodal forces $\{F\}$ and the global nodal displacements $\{d\}$ are related through use of the global stiffness matrix $[K]$ by

$$\{F\} = [K]\{d\} \quad (1.2.4)$$

Equation (1.2.4) is called the *global stiffness equation* and represents a set of simultaneous equations. It is the basic equation formulated in the stiffness or displacement method of analysis.

To obtain a clearer understanding of elements K_{ij} in Eq. (1.2.3), we use Eq. (1.2.1) and write out the expanded form of Eq. (1.2.4) as

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ \vdots \\ F_{nz} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ w_n \end{Bmatrix} \quad (1.2.5)$$

Now assume a structure to be forced into a displaced configuration defined by $u_1 = 1, v_1 = w_1 = \cdots = w_n = 0$. Then from Eq. (1.2.5), we have

$$F_{1x} = K_{11} \quad F_{1y} = K_{21}, \dots, F_{nz} = K_{n1} \quad (1.2.6)$$

Equations (1.2.6) contain all elements in the first column of $[K]$. In addition, they show that these elements, $K_{11}, K_{21}, \dots, K_{n1}$, are the values of the full set of nodal forces required to maintain the imposed displacement state. In a similar manner, the second column in $[K]$ represents the values of forces required to maintain the displaced state $v_1 = 1$ and all other nodal displacement components equal to zero. We should now have a better understanding of the meaning of stiffness influence coefficients.

Subsequent chapters will discuss the element stiffness matrices $[k]$ for various element types, such as bars, beams, plane stress, and three-dimensional stress. They will also cover the procedure for obtaining the global stiffness matrices $[K]$ for various structures and for solving Eq. (1.2.4) for the unknown displacements in matrix $\{d\}$.

Using matrix concepts and operations will become routine with practice; they will be valuable tools for solving small problems longhand. And matrix methods are crucial to the use of the digital computers necessary for solving complicated problems with their associated large number of simultaneous equations.

1.3 Role of the Computer

As we have said, until the early 1950s, matrix methods and the associated finite element method were not readily adaptable for solving complicated problems. Even though the finite element method was being used to describe complicated structures, the resulting large number of algebraic equations associated with the finite element method of structural analysis made the method extremely difficult and impractical to use. However, with the advent of the computer, the solution of thousands of equations in a matter of minutes became possible.

The first modern-day commercial computer appears to have been the UNIVAC, IBM 701, which was developed in the 1950s. This computer was built based on vacuum-tube technology.

Along with the UNIVAC came the punch-card technology whereby programs and data were created on punch cards. In the 1960s, transistor-based technology replaced the vacuum-tube technology due to the transistor's reduced cost, weight, and power consumption and its higher reliability. From 1969 to the late 1970s, integrated circuit-based technology was being developed, which greatly enhanced the processing speed of computers, thus making it possible to solve larger finite element problems with increased degrees of freedom. From the late 1970s into the 1980s, large-scale integration as well as workstations that introduced a windows-type graphical interface appeared along with the computer mouse. The first computer mouse received a patent on November 17, 1970. Personal computers had now become mass-market desktop computers. These developments came during the age of networked computing, which brought the Internet and the World Wide Web. In the 1990s the Windows operating system was released, making IBM and IBM-compatible PCs more user friendly by integrating a graphical user interface into the software.

The development of the computer resulted in the writing of computational programs. Numerous special-purpose and general-purpose programs have been written to handle various complicated structural (and nonstructural) problems. Programs such as [46–56] illustrate the elegance of the finite element method and reinforce understanding of it.

In fact, finite element computer programs now can be solved on single-processor machines, such as a single desktop or laptop personal computer (PC) or on a cluster of computer nodes. The powerful memories of the PC and the advances in solver programs have made it possible to solve problems with over a million unknowns.

To use the computer, the analyst, having defined the finite element model, inputs the information into the computer. This information may include the position of the element nodal coordinates, the manner in which elements are connected, the material properties of the elements, the applied loads, boundary conditions, or constraints, and the kind of analysis to be performed. The computer then uses this information to generate and solve the equations necessary to carry out the analysis.

1.4 General Steps of the Finite Element Method

This section presents the general steps included in a finite element method formulation and solution to an engineering problem. We will use these steps as our guide in developing solutions for structural and nonstructural problems in subsequent chapters.

For simplicity's sake, for the presentation of the steps to follow, we will consider only the structural problem. The nonstructural heat-transfer, fluid mechanics, and electrostatics problems and their analogies to the structural problem are considered in Chapters 13 and 14.

Typically, for the structural stress-analysis problem, the engineer seeks to determine displacements and stresses throughout the structure, which is in equilibrium and is subjected to applied loads. For many structures, it is difficult to determine the distribution of deformation using conventional methods, and thus the finite element method is necessarily used.

There are three primary methods that can be used to derive the finite element equations of a physical system. These are (1) the *direct method* or *direct equilibrium method for structural analysis problems*, (2) the *variational methods consisting of among the subsets energy methods and the principle of virtual work*, and (3) the *weighted residual methods*. We briefly describe these three primary methods as follows, and more details of each will be described later in this section under step 4.

Direct Methods

The direct method, being the simplest and yielding a clear physical insight into the finite element method, is recommended in the initial stages of learning the concepts of the finite element method. However, the direct method is limited in its application to deriving element stiffness matrices for one-dimensional elements involving springs, uniaxial bars, trusses, and beams.

There are two general direct approaches traditionally associated with the finite element method as applied to structural mechanics problems. One approach, called the *force*, or *flexibility, method*, uses internal forces as the unknowns of the problem. To obtain the governing equations, first the equilibrium equations are used. Then necessary additional equations are found by introducing compatibility equations. The result is a set of algebraic equations for determining the redundant or unknown forces.

The second approach, called the *displacement*, or *stiffness, method*, assumes the displacements of the nodes as the unknowns of the problem. For instance, compatibility conditions requiring that elements connected at a common node, along a common edge, or on a common surface before loading remain connected at that node, edge, or surface after deformation takes place are initially satisfied. Then the governing equations are expressed in terms of nodal displacements using the equations of equilibrium and an applicable law relating forces to displacements.

These two direct approaches result in different unknowns (forces or displacements) in the analysis and different matrices associated with their formulations (flexibilities or stiffnesses). It has been shown [34] that, for computational purposes, the displacement (or stiffness) method is more desirable because its formulation is simpler for most structural analysis problems. Furthermore, a vast majority of general-purpose finite element programs have incorporated the displacement formulation for solving structural problems. Consequently, only the displacement method will be used throughout this text.

Variational Methods

The variational method is much easier to use for deriving the finite element equations for two- and three-dimensional elements than the direct method. However, it requires the existence of a functional, that upon minimizing yields the stiffness matrix and related element equations. For structural/stress analysis problems, we can use the principle of minimum potential energy as the functional, for this principle is a relatively easy physical concept to understand and has likely been introduced to the reader in an undergraduate course in basic applied mechanics [35].

It can be used to develop the governing equations for both structural and nonstructural problems. The variational method includes a number of principles. One of these principles, used extensively throughout this text because it is relatively easy to comprehend and is often introduced in basic mechanics courses, is the theorem of minimum potential energy that applies to materials behaving in a linear-elastic manner. This theorem is explained and used in various sections of the text, such as Section 2.6 for the spring element, Section 3.10 for the bar element, Section 4.7 for the beam element, Section 6.2 for the constant strain triangle plane stress and plane strain element, Section 9.1 for the axisymmetric element, Section 11.2 for the three-dimensional solid tetrahedral element, and Section 12.2 for the plate bending element. A functional analogous to that used in the theorem of minimum potential energy is then employed to develop the finite element equations for the nonstructural problem of heat transfer presented in Chapter 13.

Another variational principle often used to derive the governing equations is the principle of virtual work. This principle applies more generally to materials that behave in a linear-elastic fashion, as well as those that behave in a nonlinear fashion. The principle of virtual work is described in Appendix E for those choosing to use it for developing the general governing finite element equations that can be applied specifically to bars, beams, and two- and three-dimensional solids in either static or dynamic systems.

Weighted Residual Methods

The weighted residual methods [36] allow the finite element method to be applied directly to any differential equation without having the existence of a variational principle. Section 3.12 introduces the Galerkin method (a very well-known residual method) for deriving the bar element stiffness matrix and associated element equations. Section 3.13 introduces other residual methods for solving the governing differential equation for the axial displacement along a bar.

The finite element method involves modeling the structure using small interconnected elements called *finite elements*. A displacement function is associated with each finite element. Every interconnected element is linked, directly or indirectly, to every other element through common (or shared) interfaces, including nodes and/or boundary lines and/or surfaces. By using known stress/strain properties for the material making up the structure, one can determine the behavior of a given node in terms of the properties of every other element in the structure. The total set of equations describing the behavior of each node results in a series of algebraic equations best expressed in matrix notation.

We now present the steps, along with explanations necessary at this time, used in the finite element method formulation and solution of a structural problem. The purpose of setting forth these general steps now is to expose you to the procedure generally followed in a finite element formulation of a problem. You will easily understand these steps when we illustrate them specifically for springs, bars, trusses, beams, plane frames, plane stress, axisymmetric stress, three-dimensional stress, plate bending, heat transfer, fluid flow, and electrostatics in subsequent chapters. We suggest that you review this section periodically as we develop the specific element equations.

Keep in mind that the analyst must make decisions regarding dividing the structure or continuum into finite elements and selecting the element type or types to be used in the analysis (step 1), the kinds of loads to be applied, and the types of boundary conditions or supports to be applied. The other steps, 2 through 7, are carried out automatically by a computer program.

Step 1 Discretize and Select the Element Types

Step 1 involves dividing the body into an equivalent system of finite elements with associated nodes and choosing the most appropriate element type to model most closely the actual physical behavior. The total number of elements used and their variation in size and type within a given body are primarily matters of engineering judgment. The elements must be made small enough to give usable results and yet large enough to reduce computational effort. Small elements (and possibly higher-order elements) are generally desirable where the results are changing rapidly, such as where changes in geometry occur; large elements can be used where results are relatively constant. We will have more to say about discretization guidelines in later chapters, particularly in Chapter 7, where the concept becomes quite significant.